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Modulation parameters in incommensurate modulated structures with inflation symmetry

L Elcoro†, I Etxebarria‡ and J M Perez-Mato†

† Departamento de Física de la Materia Condensada, Facultad de Ciencias, Universidad del País Vasco, Apdo 644, 48080 Bilbao, Spain

‡ Fisika Aplikatua II Saila, Zientzia Fakultatea, Euskal Herriko Unibertsitatea, P K 644, 48080 Bilbao, Spain

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Abstract. Inflation symmetry is one of the peculiar features of the diffraction pattern of a quasicrystal. However, it is not an exclusive property of quasicrystalline structures and it may be present in incommensurately modulated structures, as shown recently in the Al–Mg system (Donnadieu P *et al* 1996 *J. Physique* I **6** 1153–64). The conditions that a single modulation parameter of an incommensurate structure must fulfil in order to have inflation symmetry are determined. Although the number of possible distinct inflation-symmetric quasilattices is infinite, from physical/experimental arguments it can be concluded that, in practice, only a few of them can be experimentally observed, the reported phase of the Al–Mg system being one of these particular cases. A quantitative criterion to classify the modulation parameters that give rise to quasilattices with observable inflation symmetry is proposed. The generalization of the analysis of incommensurate structures with more than one single modulation parameter is also discussed. Finally, the inflation parameters of diffraction patterns with rotational point groups of finite order, C_N, are compared with the parameters of the one-dimensional case.

1. Introduction

It is very well known that the reciprocal quasilattice of a quasicrystal possess scaling or inflation symmetry (IS). This means that if all the vectors of the quasilattice are multiplied by a constant factor greater than one, the so-called *inflation parameter*, the quasilattice remains unchanged. On the other hand, the IS of incommensurate modulated structures (IMS) has scarcely been investigated due to the lack of experimental evidence of this property in their diffraction patterns. Janner, Ascher and Janssen [1–5] studied the presence of IS in IMS with one modulation parameter and determined the form of all the possible inflation parameters using group theory. They found that the number of quasilattices with IS is infinite. The presence of IS in more exotic systems such as snow flakes has also been studied by Janner [6].

Recently, a new metastable phase in the Al–Mg system with very odd properties [7] has been reported. At 61 at% Al the diffraction pattern of rapidly solidified samples exhibits cubic rotational symmetry and IS at the same time. As the IS has been considered to be a characteristic of quasicrystalline structures, the new metastable phase was called ‘cubic quasicrystal’. In a previous work [8], we showed that this phase can be interpreted as an *ordinary* IMS, and the six-dimensional superspace group which describes the structure was determined. The observed IS can be attributed to the specific value of the modulation parameter q, which within experimental resolution equals $2-\sqrt{3}$, while the inflation factor of the reciprocal quasilattice is $\alpha = 2+\sqrt{3}$.

In the present work we study the possible inflation parameters of an IMS with a single modulation. We also derive the possible modulation parameters which give rise to different quasilattices with IS. The use of physical/experimental arguments allow us to 'predict' the quasilattices with IS that are most likely to be experimentally observed. The way to generalize to IMS with more than one modulation parameter is also pointed out. Finally, we show in a simple way that the inflation parameters in rotational point groups of finite order, C_N, are a subset of the parameters obtained in one dimension.

2. Inflation symmetry in incommensurate modulated structures

A non-weighted quasilattice, i.e. a quasilattice with no consideration of the associated diffraction intensities, will have IS if after the multiplication of all the vectors by a constant factor, the following two conditions are satisfied:

Condition 1. All the vectors in the new quasilattice (the 'inflated one') are in the first quasilattice.

Condition 2. Every vector in the original quasilattice belongs to the inflated quasilattice.

Periodic lattices fulfil the first condition, the constant factor being an integer, but do not fulfil the second condition, thus, they do not possess that symmetry. In this section we analyse the conditions that the modulation parameters of an IMS must fulfil in order to have IS when the modulation vectors are parallel to rational directions of the sublattice of main reflections of the diffraction pattern.

2.1. One-dimensional quasilattices with one single modulation parameter

Let $\boldsymbol{k}$ be the basis vector which indexes all the main reflections in a one-dimensional quasilattice of an IMS and $\boldsymbol{q} = q\boldsymbol{k}$ is the modulation vector necessary to index the satellites. We can choose $\boldsymbol{k}$ and $\boldsymbol{q}$ as scalars with $|\boldsymbol{k}| = 1$, and any reflection may be expressed as:

$$H = n + mq \qquad \text{with } n, m \in Z. \tag{1}$$

If this line of Bragg points has IS, there is an inflation parameter α so that $H' = \alpha H$ belongs to the quasilattice (1) for every H. Taking particular values for n and m ($n = 1$, $m = 0$ and $n = 0$, $m = 1$) it is straightforward to demonstrate that α and q^2 must be rational linear combinations of l and q, so that the modulation parameter q and the inflation parameter α satisfy a second degree equation with rational coefficients. Therefore, the most general forms of the modulation parameter and the inflation factor are:

$$q = q_l \pm \sqrt{q_2} \text{ and } \alpha = q_3 \pm q_4\sqrt{q_2} \tag{2}$$

where q_1, q_2, q_3 and q_4 are rational numbers, and q_2 does not have an integer square root. Therefore, both q and α are quadratic integer numbers. For a general reflection (1), condition 1 implies that

$$H' = \alpha H = \alpha(n + mq) = n' + m'q \qquad n, m, n', m' \in Z. \tag{3}$$

As equation (3) must be fulfilled for any n and m, from (2) it can be shown that the following relations must be satisfied,

$$\begin{aligned} &q_3 - q_1q_4 = n_1 \qquad q_4(q_2 - q_1^2) = n_3 \\ &q_4 = n_2 \qquad q_3 + q_1q_4 = n_4 \end{aligned} \tag{4}$$

and relation (3) can be set in matrix form,

$$\begin{pmatrix} n' \\ m' \end{pmatrix} = \begin{pmatrix} n_1 & n_3 \\ n_2 & n_4 \end{pmatrix} \times \begin{pmatrix} n \\ m \end{pmatrix}. \tag{5}$$

On the other hand, condition 2 implies that the inverse of the 2×2 matrix of equation (5) must be an integer, so, the determinant of the matrix is ± 1,

$$n_1 n_4 - n_2 n_3 = \pm 1 \tag{6}$$

and it belongs to the group of integer 2×2 matrices with determinant ± 1, $GL(2, Z)$. Therefore, the modulation parameter q and the inflation parameter α may be expressed as:

$$q = \frac{n_4 - n_1 \pm \sqrt{4n_2 n_3 + (n_4 - n_1)^2}}{2n_2} \tag{7}$$

$$\alpha = \frac{n_1 + n_4 \pm \sqrt{4n_2 n_3 + (n_4 - n_1)^2}}{2} \tag{8}$$

where the four integers are related by equation (6) and the square root is not an integer. Equations (7) and (8) constitute the general forms for the modulation and inflation parameters in our special case. Using equation (6) the inflation factor becomes:

$$\alpha = \frac{N \pm \sqrt{N^2 \pm 4}}{2} \qquad \text{with } N = n_1 + n_4. \tag{9}$$

This last result has already been obtained by Janner and Ascher [1, 2]. The α parameter satisfies the relation,

$$\alpha^2 = \pm 1 + (n_1 + n_4)\alpha \tag{10}$$

depending on the two possibilities in equation (6). Therefore, α is a quadratic unit number (positive or negative). All the possibilities of equations (7) and (8) are solutions of the $\pm$Pell equation. The two possibilities for the sign in equation (8) correspond to a pair of conjugate values which satisfy $\alpha_+ \alpha_- = 1$. Therefore, one of these values is the inflation parameter of the quasilattice and the other is the deflation parameter. The general expressions (7) and (8) derived here contain additional information, as we will see below.

According to equation (9), the set of possible inflation parameters α is infinite, and therefore the number of possible types of IS quasilattices is also infinite. However, for a given value of $N = n_1 + n_4$ (and therefore for a single inflation parameter α) there are also *different* quasilattices that correspond to different possibilities for the n_1, and n_4 integers, keeping N constant (10). So, the value of the inflation factor restricts the possible values of the modulation parameter, but it does not determine it.

Although any combination of four integers that satisfy equation (6) is possible, not all of them are independent. For example, different combinations associated with the same inflation factor α and different modulation parameters which differ by an integer are equivalent. They correspond to different elections of the modulation vector to index the same quasilattice. The same is true when the sum of the two modulation parameters is an integer. As a general rule, to each combination of integers we will assign not the q value calculated directly from (7), but the equivalent value in the range $0 < q < 0.5$. On the other hand, it may happen that two sets of four integers generate equivalent modulation vectors (their sum or subtraction is an integer), but different inflation factors. In this case, as the quasilattice is the same in both cases, it can be proved that one of the inflation factors is an integer power of the other. This reflects the trivial fact that for an inflation factor α of a quasilattice, for any integer n positive or negative, α^n will also be an inflation parameter. Among all the equivalent choices, we will take the value closest to and greater than one. Note that values of the inflation parameter in the range $0 < \alpha < 1$ correspond, in fact, to 'deflation' parameters and negative values of α give rise to inflation/deflation plus inversion.

Finally, for any combination of integers we can always choose the positive sign of the square root in equation (7), because there is another combination ($n'_1 = n_4$, $n'_2 = -n_2$, $n'_3 = -n_3$, $n'_4 = n_1$) that changes only the sign of the square root, leaving the inflation parameter unchanged.

Even if we take into account all these considerations, the number of combinations of four integers which satisfy equation (6) and generate non-equivalent quasilattices with IS is infinite, and the values of the possible modulation vectors producing IS densely fill the range between 0 and 0.5. This means that, mathematically, *any* value of the modulation vector of an IMS is infinitely close to a modulation parameter described by equation (7) with a certain set of four integers. Thus, there is always a quasilattice with IS that cannot be experimentally distinguished from the quasilattice of the real structure. However, to our knowledge there is only one experimentally observed IMS where IS has been claimed [7]. The reason is that, in practice, the observation of IS features in a diffraction diagram is implicitly considered to be the set of discrete Bragg peaks, and not the dense set of points of the corresponding quasilattice. Note that for a general reflection, $H = n + mq$ the set of four indices of the matrix in equation (5) gives the indices of the reflection $H' = \alpha(n + mq)$ related to the first set by IS. If the four integers are large, the indices of H' will be large compared to n and m, and experimentally this satellite reflection will not be observable. Moreover, the inflation factor will also be large and the related reflection will be far from the first value, making it difficult to visualize the IS. These facts can be used to establish a hierarchy between quasilattices according to the degree of practical observability of the IS: the quasilattices with low values of the integers n_i will be the best candidates to be observed and a criterion of observability can be defined in terms of the sum of their absolute values:

$$M = \sum_{i=1}^{4} |n_i| \tag{11}$$

and it can be concluded that the diffraction diagrams will apparently exhibit less IS visual features the larger the value of M associated with the modulation parameter.

Considering this criterion of observability, we have made a systematic search of all the sets of four integers that satisfy equation (6) with the smallest M values, taking into account the previous arguments about the equivalence of some of the sets. All the possibilities for $M < 11$ are shown in table 1. Each row corresponds to a different quasilattice with IS. In the first four columns, the four integers of the matrix (5) are indicated, the fifth is the sum of their absolute values (11), the sixth column represents the modulation parameter given by the four integers (7) and its equivalent value in the range $0 < q < 0.5$, and in the last column the associated inflation parameter (8) is given. As we have mentioned before, some of these quasilattices have the same value for the inflation parameter, but as the modulation vector of one of them cannot be indexed by means of the basis vectors of the other quasilattice, the quasilattices are different. In some of these cases one quasilattice is a subset of the other (for example, the quasilattice of row 3 is a subset of the quasilattice of row 18), but in other cases there is not such a relation (quasilattices of rows 13 and 15). As it has been stressed before, these inequivalences are associated with the different possibilities for the n_1 and n_4 integers, which give rise to the same quadratic equation (10). According to the criterion explained above, among all the possibilities, the first rows in table 1 are the main candidates to show IS in practice, and in fact the Al–Mg structure with cubic point group symmetry and reported IS corresponds to row 5 of table 1 [8]. Note also that the modulation parameter $(3 - \sqrt{5})/2$ with the third lowest value of M is equivalent to the golden mean $(1 + \sqrt{5})/2$ appearing in

Table 1. Modulation parameters of an incommensurately modulated structure which give rise to inflation symmetry in the reciprocal quasilattice. In the first four columns, the four integers of matrix (5) are indicated. The fifth column corresponds to the value of (11) which serves as a criterion to order the resulting quasilattices. The sixth column indicates the modulation parameter given by (7) and the equivalent parameter in the range $0 < q < 0.5$ when they are different. The last column gives the inflation factor associated with the quasilattice (equation (8)).

n_1	n_2	n_3	n_4	$\sum \lvert n_i \rvert$	Modulation parameter, q	Equivalent parameter in $0 < q < 0.5$	Inflation parameter, α
2	1	1	0	4		$\sqrt{2} - 1 \approx 0.4142$	$\sqrt{2} + 1 \approx 2.4142$
3	1	1	0	5		$\frac{\sqrt{13}-3}{2} \approx 0.3028$	$\frac{\sqrt{13}+3}{2} \approx 3.3028$
2	−1	1	−1	5		$\frac{3-\sqrt{5}}{2} \approx 0.3820$	$\frac{\sqrt{5}+1}{2} \approx 1.6180$
4	1	1	0	6		$\sqrt{5} - 2 \approx 0.2361$	$\sqrt{5} + 2 \approx 4.2361$
4	−1	1	0	6		$2 - \sqrt{3} \approx 0.2679$	$\sqrt{3} + 2 \approx 3.7321$
3	−2	1	−1	7		$\frac{2-\sqrt{2}}{2} \approx 0.2929$	$\sqrt{2} + 1 \approx 2.4142$
3	2	1	1	7		$\frac{\sqrt{3}-1}{2} \approx 0.3660$	$\sqrt{3} + 2 \approx 3.7321$
5	1	1	0	7		$\frac{\sqrt{29}-5}{2} \approx 0.1926$	$\frac{\sqrt{29}+5}{2} \approx 5.1926$
5	−1	1	0	7		$\frac{5-\sqrt{21}}{2} \approx 0.2087$	$\frac{\sqrt{21}+5}{2} \approx 4.7913$
2	3	1	1	7		$\frac{\sqrt{13}-1}{6} \approx 0.4343$	$\frac{\sqrt{13}+3}{2} \approx 3.3028$
1	3	1	2	7	$\frac{\sqrt{13}+1}{6}$	$\equiv \frac{5-\sqrt{13}}{6} \approx 0.2324$	$\frac{\sqrt{13}+3}{2} \approx 3.3028$
4	3	1	1	9		$\frac{\sqrt{21}-3}{6} \approx 0.2638$	$\frac{\sqrt{21}+5}{2} \approx 4.7913$
3	4	1	1	9		$\frac{\sqrt{5}-1}{4} \approx 0.3090$	$\sqrt{5} + 2 \approx 4.2361$
1	4	1	3	9	$\frac{\sqrt{5}+1}{4}$	$\equiv \frac{3-\sqrt{5}}{4} \approx 0.1910$	$\sqrt{5} + 2 \approx 4.2361$
2	5	1	2	10		$\frac{\sqrt{5}}{5} \approx 0.4472$	$\sqrt{5} + 2 \approx 4.2361$
4	5	1	1	11		$\frac{\sqrt{29}-3}{10} \approx 0.2385$	$\frac{\sqrt{29}+5}{2} \approx 5.1926$
1	5	1	4	11	$\frac{\sqrt{29}+3}{10}$	$\equiv \frac{7-\sqrt{29}}{10} \approx 0.1615$	$\frac{\sqrt{29}+5}{2} \approx 5.1926$
3	−5	1	−2	11		$\frac{5-\sqrt{5}}{10} \approx 0.2764$	$\frac{\sqrt{5}+1}{2} \approx 1.6180$
3	5	1	2	11		$\frac{\sqrt{21}-1}{10} \approx 0.3583$	$\frac{\sqrt{21}+5}{2} \approx 4.7913$
2	5	1	3	11	$\frac{\sqrt{21}+1}{10}$	$\equiv \frac{9-\sqrt{21}}{10} \approx 0.4417$	$\frac{\sqrt{21}+5}{2} \approx 4.7913$
5	4	1	1	11		$\frac{\sqrt{2}-1}{2} \approx 0.2071$	$2\sqrt{2} + 3 \approx 5.8284$
5	−3	2	−1	11		$\frac{3-\sqrt{3}}{3} \approx 0.4226$	$\sqrt{3} + 2 \approx 3.7321$
5	3	2	1	11		$\frac{\sqrt{10}-2}{3} \approx 0.3874$	$\sqrt{10} + 3 \approx 6.1623$
1	3	2	5	11	$\frac{\sqrt{10}+2}{3}$	$\equiv \frac{4-\sqrt{10}}{3} \approx 0.2792$	$\sqrt{10} + 3 \approx 6.1623$

the usual definition of the Fibonacci chain and the length scales of decagonal and icosahedral quasicrystals (see, for example, [3]).

The physical meaning of the IS is still an open question. Is the presence of IS a symmetry feature of particular thermodynamic phases in the phase diagram or is the observation of IS merely an accident due to the continuous variation in the phase diagram of the modulation parameter? It is significant that the intensity of satellite reflections of the previously mentioned cases of IMS with IS is very high and that they are present up to a high order, in comparison with modulated structures with different modulation parameters in the same alloy [7]. In order to elucidate this question, it would be very interesting to look for IMS with a modulation parameter given in the first rows of table 1, check the region of stability of the system and analyse the intensity and order of the satellite reflections. If there was a significant difference in the intensities and number of observed satellite reflections with respect to similar IMS with modulation parameters far from the values of table 1, then it would indicate that the IS resulting from these particular values of the modulation parameter has some physical meaning and is not merely a mathematical curiosity.

2.2. One-dimensional quasilattices with more than one modulation parameter

When two modulation parameters are necessary to index a line of the diffraction pattern, say q and r (both mutually incommensurate), the two conditions previously mentioned would require that the inflation parameter and the three quantities, q^2, r^2 and qr, be rational linear combinations of 1, q and r. It can be deduced that q and r satisfy third degree equations with rational coefficients, i.e. they are cubic integer numbers. The matrices relating the indices of a reflection and its (inflation) symmetry-related one belong to the group of integer matrices 3×3 with determinant ± 1, $GL(3, Z)$. When more than two modulation vectors are necessary, the equations which would satisfy the modulation parameters would be of a larger degree, i.e. they are integer numbers of higher order and the matrix would be of a higher dimension, its determinant always being ± 1. In this case, there would be more than one single inflation parameter which cannot be expressed as integer powers of one of them. Although mathematically possible, it would be rather difficult to observe a diffraction pattern with this property.

2.3. Inflation symmetry in three-dimensional incommensurate modulated structures

In physical three-dimensional systems, the IS can be present in one, two or three dimensions. For example, the diffraction pattern of a structure, in which the atoms are arranged periodically in two dimensions and following a Fibonacci chain in the third dimension, will exhibit IS in this direction. In this case, the third component of all reciprocal vectors must be multiplied by the inflation factor to generate the inflated quasilattice, the other two components remaining unchanged. IS can be present in a plane, being the quasilattice in the third direction periodic, as in decagonal, octagonal and dodecagonal quasicrystals. The two components of the diffraction vectors parallel to the quasiperiodic plane must be multiplied by the inflation parameter in order to obtain the inflated quasilattice. Finally, the diffraction pattern of an icosahedral quasicrystal possesses IS in the whole reciprocal space. In this section we are interested in IMS with IS in more than a single direction. As in the previous sections, the modulation vectors are considered to be parallel to the vector basis of the set of main reflections. Moreover, we will restrict ourselves to operations that are merely a dilation (we do not consider more general operations like a rotation combined with inflation). Within these conditions, once the possible modulation parameters in one dimension have been determined, the point symmetry of the IMS forces the modulation parameter to take the same value in all symmetry-related directions. For example, in the case of a cubic crystal, rotational symmetry forces the modulation parameter to be the same in the three orthogonal directions. This is precisely the case in the Al–Mg structure previously mentioned [7]. In tetragonal, hexagonal or trigonal systems the IS can be present in one direction (parallel to the four-, six- or three-fold axis), in the plane perpendicular to this direction or in both of them. In general, the inflation parameter can take different values in the plane and in the axis direction. In the rest of the systems, as the point group does not mix different directions, IS can be present in just one dimension, in two dimensions or even in three dimensions, with the inflation parameters generally different for each direction.

3. IS in quasilattices with rotational point group of finite order, C_N

In this section we analyse the IS of quasilattices having within their point group a rotational symmetry of finite order C_N. It is verified in a straightforward manner that the inflation parameters and incommensurate parameters present in the directions parallel to each vector basis are included among the possibilities of the one-dimensional case. More formal and general analyses of the subject can be found in [9–11].

As stressed above, in all the cases, the modulation parameter and the inflation factor must be solutions of equations of different degrees. We will see in the following that in the quasilattices with point groups having an N-fold axis, this condition is forced by the point group symmetry.

In the diffraction pattern of quasiperiodic structures with point group C_N, the N-fold axis relates reflections in a star of N ($2N$) vectors that subtend angles of $2\pi/N$ ($2\pi/(2N)$) for N even (odd). The number of rationally independent vectors in that star is the rank of the quasilattice. Moreover, if we denote these vectors as $\boldsymbol{k}_i$ ($i = 1, \ldots, N$ and their opposites if N is odd, and ordered from lower to higher angles with respect to $\boldsymbol{k}_1$) it is clear that the set of vectors ($\boldsymbol{k}_i, \boldsymbol{k}_{i+1} + \boldsymbol{k}_{i-1}, \boldsymbol{k}_{i+2} + \boldsymbol{k}_{i-2}, \ldots$) are all parallel, and that the relations between their modulus and the modulus of $\boldsymbol{k}_i$ are

$$1, 2\cos(2\pi/N), 2\cos(4\pi/N), \ldots \tag{12}$$

If the number of rationally independent numbers among them is L, $L - 1$ is the number of modulation vectors in a single line of the diffraction pattern parallel to $\boldsymbol{k}_i$. For $N = 1$, 2, 3, 4, 6 there is only one independent value and the diffraction pattern is periodic in all lines. This corresponds to the crystalline case and there is no IS. If $N = 5$, 8, 10, 12 there are two rationally independent numbers, 1 and $2\cos(2\pi/N)$. Therefore, every reflection in one line can be indexed by two mutually rationally independent vectors, being the modulation parameter $2\cos(2\pi/N)$ (or an integer $\pm$ that value). If $N = 7, 9, \ldots$ there are three rationally independent values. In any case, $2\cos(2\pi/N)$ satisfies an equation of degree N with rational coefficients, which can be factorized. Depending on the factorization, the degree of the irreducible polynomial equations which satisfies $2\cos(2\pi/N)$ will be of a different value. In order to calculate the polynomial in each case, the following trigonometric relations must be used:

$$\begin{aligned} \cos N\theta &= \cos(N-1)\theta\cos\theta - \sin(N-1)\theta\sin\theta \\ \sin N\theta &= \sin(N-1)\theta\cos\theta + \cos(N-1)\theta\sin\theta. \end{aligned} \tag{13}$$

For successive values of N the relations represent two recurrent sequences. We can expand $\cos N\theta$ as a polynom of degree N in $\cos\theta$, and put $\sin N\theta$ as $\sin\theta$ times a polynom of degree $N-1$ in $\cos\theta$. Equating the first polynomial to 1, the roots of that equation will be $\cos 2\pi/N$, $\cos 2(2\pi/N), \ldots, \cos N(2\pi/N) = 1$. As this latter trivial solution is always present for all N, the equation can always be decomposed into two equations of degree 1 and ($N - 1$), and the latter can generally be reduced further. In crystalline cases, $N = 1$, 2, 3, 4, 6, the equation can be decomposed into first degree equations whose roots are all rational. For $N = 5$, 8, 10, 12 the equation is decomposable into equations of degree 1 and 2, whose solutions have the form of q in (2), so that, in one line there is IS and both the modulation and the inflation parameter, must be given in table 1. It can be checked that $N = 5$, 10 cases correspond to the third row (golden mean), $N = 8$ to the first row and $N = 12$ to the fifth row.

The $N = 7$ case is quite different. Two modulation vectors are necessary to index all reflections in a line of the diffraction pattern. The seventh degree equation obtained can be decomposed into one equation of first degree and two identical equations of third degree, $8x^3 + 4x^2 - 4x - 1 = 0$. The three solutions of this equation are $\cos 2\pi/7$, $\cos 4\pi/7$ and $\cos 6\pi/7$, but the last one is an integer linear combination of the other two solutions and 1. Therefore, the two incommensurate parameters in equation (12), $q = 2\cos 2\pi/7$ and $r = 2\cos 4\pi/7$, are solutions of a third degree equation and q^2, r^2 and qr are integer linear combinations of 1, q and r. These modulation parameters cannot be described in the form (2) and are therefore not present in table 1, which is restricted to the case of a single modulation parameter per direction in the reciprocal space. As it has been pointed out in section 2.2 there is a 3×3 integer matrix belonging to $GL(3, Z)$ which relates a reflection with its 'inflated' symmetry-related one. It can be proved that both q and r are inflation parameters of the quasilattice, and one of them is not an integer power of the other. The same happens with the $N = 9$ quasilattice, whose irreducible equations are of third degree. The cases $N = 7$ and $N = 9$ have been discussed by Barache [9] and Gazeau [11].

Therefore, the inflation parameters appearing in observed quasicrystals are all included in table 1. They correspond to IS with low parameter M and hence are observable in the diffraction diagram. On the other hand, it is significant that the rotational symmetries as $N = 7, 9$, not experimentally observed, imply IS with two independent inflation parameters.

4. Conclusions

The modulation parameters and inflation factors associated with an IMS with one single modulation have been calculated. They are infinite but physical/experimental reasons can be used to classify them according to a criterion of practical observability. The modulation parameter of the 'cubic quasicrystal' recently observed in the Al–Mg diagram is, in principle, one of the most likely to be observed. Until now, it is an open question as to whether inflation symmetry has physical meaning or if it is merely a mathematical curiosity due to the continuity of the modulation parameter as a function of composition or temperature. Table 1 can help to look for quasilattices with such a symmetry and elucidate this question. The way to generalize to IMS with more than a modulation in one direction has also been pointed out. Finally, rotational groups of any order, in which the inflation symmetry is forced by the point group, have been analysed.

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